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Janossy densities, multimatrix spacing distributions and Fredholm resolvents

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Abstract

A simple proof is given for a generalized form of a theorem of Soshnikov. The latter states that the Janossy densities for multilevel determinantal ensembles supported on measurable subspaces of a set of measure spaces are constructed by dualization of bases on dual pairs of N-dimensional function spaces with respect to a pairing given by integration on the complements of the given measurable subspaces. The generalization extends this to dualization with respect to measures modified by arbitrary sets of weight functions.

1. Multilevel determinantal ensembles.

Let $\{(\Gamma_j, d\mu_j)\}_{j=1\dots m}$ be a set of measure spaces and $\{\tilde{H}_j := L^2(\Gamma_j, d\mu_j)\}_{j=1\dots m}$ the Hilbert spaces of square integrable functions on them. Let $\{f_a\}_{a=1\dots N}$ and $\{h_a\}_{a=1\dots N}$ be bases for a pair of N-dimensional subspaces $H_1 \subset \tilde{H}_1$, $H^m \subset \tilde{H}_m$, respectively. Suppose we are given $m-1$ functions $\{g_{j+1,j}\}_{j=1\dots m-1}$ on the product spaces $\Gamma_{j+1} \times \Gamma_j$, such that the corresponding integral operators

$$\begin{aligned} g_{j+1,j} &: \hat{H}_j \longrightarrow \hat{H}_{j+1} \\ g_{j+1,j}(f)(x_{j+1}) &:= \int_{\Gamma_j} g_{j+1,j}(x_{j+1}, x_j) f(x_j) d\mu_j(x_j), \end{aligned} \tag{1.1}$$

together with their transposes

$$\begin{aligned} g^*_{j+1,j} &: \hat{H}_{j+1} \longrightarrow \hat{H}_j \\ g^*_{j+1,j}(f)(x_j) &:= \int_{\Gamma_{j+1}} g_{j+1,j}(x_{j+1}, x_j) f(x_{j+1}) d\mu_j(x_{j+1}), \end{aligned} \tag{1.2}$$

are well defined injective maps on a sequence of dense subspaces $\hat{H}_1 \subset \widetilde{H}_1$, $\hat{H}_2 = g_{21}(\hat{H}_1) \subset \widetilde{H}_2$, $\dots$, $\hat{H}_m = g_{m,m-1}(\hat{H}_{m-1}) \subset \widetilde{H}_m$, as are their composites:

$$g_{kj} := g_{k,k-1} \circ \dots \circ g_{j+1,j}. \tag{1.3}$$

We assume that $H_1 \subset \hat{H}_1$, $H^m \subset \widetilde{H}_m$, and denote the respective images as $H_2 := g_{21}(H_1), \dots,$ $H_m := g_{m,m-1}(H_{m-1})$ and $H^{m-1} := g^*_{m,m-1}(H^m), \dots,$ $H^1 := g^*_{2,1}(H^2)$. Assuming furthermore that the $N \times N$ matrix:

$$A_{ab} := \int_{\Gamma_1} f_a(x_1) g^*_{m1}(h_b)(x_1) d\mu_1(x_1) = \int_{\Gamma_m} g_{m1}(f_a)(x_m)(h_b)(x_m) d\mu_m(x_m) \tag{1.4}$$

is nonsingular, it follows that the pairs of spaces $\{H_j, H^j\}$ may be viewed as mutually dual, and identified through the $d\mu_j$ integration pairings. Making a PLU decomposition of A

$$A := PLU, \tag{1.5}$$

where L and U are, respectively, lower and upper triangular matrices, normalized, e.g., with equal diagonal entries (unique up to 2^N sign ambiguities on the diagonal), and P a permutation matrix, we may form bases:

$$\psi^{(1)}_a := \sum_{b=1}^N (PL)^{-1}_{ab} f_b \qquad \phi^{(m)}_a := \sum_{b=1}^N U^{-1}_{ba} h_b \tag{1.6}$$

for the spaces H_1 and H^m, respectively, as well as for the sequence of dual spaces $\{H_j, H^j\}$ through composition with g_{j1} and g^*_{mj}:

$$\{\psi^{(j)}_a := g_{j1}(\psi^{(1)}_a)\}_{a=1\dots N}, \qquad \{\phi^{(j)}_a := g^*_{mj}(\phi^{(m)}_a)\}_{a=1\dots N}. \tag{1.7}$$

These are, by construction, mutually dual,

$$\int_{\Gamma_j} \psi^{(j)}_a(x_j) \phi^{(j)}_b(x_j) d\mu_j(x_j) = \delta_{ab}. \tag{1.8}$$

An example of the above construction consists of choosing the sets Γ_j to be intervals on the real line, with Lebesgue measure, and the functions f_a, h_a and $\{g_{j,j-1}\}$ of the form:

$$\begin{gathered} f_a(x_1) := x_1^{a-1} e^{-\frac{1}{2}V_1(x_1)}, \quad h_a(x_m) := x_m^{a-1} e^{-\frac{1}{2}V_m(x_m)}, \\ g_{j,j-1}(x_j, x_{j-1}) := e^{x_{j-1}x_j - \frac{1}{2}(V_{j-1}(x_{j-1}) + V_j(x_j))}, \end{gathered} \tag{1.9}$$

where the $V_j(x_j)$'s are suitably defined functions for which the integrals involved are convergent (e.g., polynomials of even degree with real, positive leading coefficients). This case arises in the study of random multimatrix chains models [EM, BEH1, BEH2]. The reduced probability density for eigenvalues $\{x^{(j)}_a\}_{j=1\dots m, a=1\dots N}$ is given by the determinantal formula

$$P^m_N(x^{(j)}_a) = \frac{1}{Z_{N,m}} \det(\psi^{(1)}_a(x^{(1)}_b)) \det(\phi^{(m)}_a(x^{(m)}_b)) \prod_{j=1}^{m-1} \det(g_{j+1,j}(x^{(j+1)}_a, x^{(j)}_b)) \tag{1.10}$$

where the partition function $Z_{N,m}$ is defined so as to normalize this to a probablility measure.

More generally, let us suppose that the functions $f_a, h_a, g_{j,j-1}$ are chosen so that the expression (1.10) defines a probablity measure on the ensemble $\prod_{j=1}^m (\Gamma_j)^N$. Such ensembles are sometimes referred to as "determinantal ensembles" [BS, S]; other examples include, e.g., "polynuclear growth" models [PS, J].

Define the following set of functions on the product spaces $\Gamma_i \times \Gamma_j$:

$$K_{ij}(x_i, x_j) := \sum_{a=1}^{N} \psi_a^{(i)}(x_i)\phi_a^{(j)}(x_j) \tag{1.11}$$

and

$$\check{K}_{ij}(x_i, x_j) := K_{ij}(x_i, x_j) - g_{ij}(x_i, x_j), \tag{1.12}$$

where $g_{ij}(x_i, x_j) := 0$ if $i \leq j$. It may then be shown [EM] that that the probability density (1.10) may equivalently be expressed in the form:

$$P_{N,m}(x_a^{(j)}) = \det(\check{K}_{ij}(x_a^{(i)}, x_b^{(j)})) \ , \tag{1.13}$$

where $\check{K}_{ij}(x_a^{(i)}, x_b^{(j)})$ is viewed as the $((i,a),(j,b))$ element of a matrix of dimension $Nm \times Nm$, labelled by pairs of double indices $1 \leq i, j \leq m, \ 1 \leq a, b \leq N$. By integrating (1.13) over a part of the variables, it follows [EM] that the correlation function giving the probability density for finding k_j elements in Γ_j at the points $\{x_1^{(j)}, \dots x_{k_j}^{(j)}\}$ for $j = 1 \dots m$ is similary expressed by the $\sum_{j=1}^m k_j \times \sum_{j=1}^m k_j$ determinant

$$\rho_{k_1, \dots k_m}(\{x_1^{(j)}, \dots x_{k_j}^{(j)}\}_{j=1\dots m}) = \det(\check{K}_{ij}(x_a^{(i)}, x_b^{(j)}))|_{\substack{1 \leq a \leq k_i \\ 1 \leq b \leq k_j}} \ . \tag{1.14}$$

The functions K_{ij} may also be viewed as kernels of integral operators

$$\begin{aligned} K_{ij} : \hat{H}_j &\longrightarrow \hat{H}_i \\ K_{ij}(f)(x_i) &:= \int_{\Gamma_j} K_{ij}(x_i, x_j) f(x_j) d\mu_j(x_j) \end{aligned} \tag{1.15}$$

which, again, map the finite dimensional spaces $\{H_j\}$ to each other. Note that the various K_{ij}'s may all be obtained from K_{1m} by composition on the left and right with operators g_{ij}:

$$K_{ij} = g_{i1} \circ K_{1m} \circ g_{mj}, \tag{1.16}$$

and that, in particular, adjacent ones are related by

$$K_{ij} = g_{i,i-1} \circ K_{i-1,j} = K_{i,j+1} \circ g_{j+1,j}. \tag{1.17}$$

It follows from (1.8) that, when restricted to H_j and H^j, K_{jj} and K^*_{jj} act as identity operators

$$K_{jj}(\psi) = \psi, \quad \psi \in H_j, \qquad K^*_{jj}(\phi) = \phi, \quad \phi \in H^j. \tag{1.18}$$

In the following, we denote by K, g and $\check{K}$ the $m \times m$ matrices of integral operators acting on the direct sum

$$\hat{H} := \oplus_{j=1}^m \hat{H}_j, \tag{1.19}$$

with matrix entries K_{ij}, g_{ij} and $\check{K}_{ij}$ acting on the component spaces $\hat{H}_j$.

Let $\{I_j \subset \Gamma_j\}_{j=1\dots m}$ be measurable subsets of the spaces Γ_j, and let χ_j denote the characteristic function of $I_j \subset \Gamma_j$. Denote by χ_I the direct sum of these functions, viewed as an operator on $\hat{H}$, acting by multiplication by the various χ_j's on the component spaces $\hat{H}_j$'s, and assume that this leaves $\hat{H}$ invariant. Let

$$\check{K}^{\chi_I} := \check{K} \circ \chi_I \tag{1.20}$$

denote the composition of these operators, also a matrix integral operator, and let

$$R^{\chi_I} := (\mathbf{1} - \check{K}^{\chi_I})^{-1} \circ \check{K}^{\chi_I} \tag{1.21}$$

be its Fredholm resolvent. Then the matrix components of this resolvent, denoted R_{ij}, are operators with integral kernels $R_{ij}(x_i, x_j)$. These also define certain correlation functions, on the product $\prod_{j=1}^m I_j^N$, the so-called *Janossy* densities, given by a formula similar to (1.14), namely:

$$\rho^I_{k_1\dots k_m}(\{x_1^{(j)}, \dots x_{k_j}^{(j)}\}_{j=1\dots m}) = C_I^{N,m} \det(R_{ij}(x_a^{(i)}, x_b^{(j)}))|_{\substack{1\le a\le k_i \\ 1\le b\le k_j}}\ , \tag{1.22}$$

where the normalization constant $C_I^{N,m}$ is defined to be the Fredholm determinant:

$$C_I^{N,m} := \det(\mathbf{I} - \check{K}^{\chi_I}) \tag{1.23}$$

(which equals the probablity, under the original distribution (1.10), of having no elements within the subset $I = \prod_{j=1}^m I_j^N$). The correlation functions (1.22) give the probability density of finding, for $j = 1 \dots m$, exactly k_j elements in I_j at the points $\{x_1^{(j)}, \dots x_{k_j}^{(j)}\}_{j=1\dots m}$.

A theorem of Soshnikov [S] expresses this distribution in terms of a new set of functions $\{\widetilde{\psi}^{(j)}, \widetilde{\phi}^{(j)}, \widetilde{g}_{ij}\}$, analogous to $\{\psi^{(j)}, \phi^{(j)}, g_{ij}\}$. To define these functions, one begins by defining, as in (1.3), integral operators

$$\widetilde{g}_{kj} := g_{k,k-1} \circ_{I^c_{k-1}} \dots \circ_{I^c_{j+1}} g_{j+1,j} \tag{1.24}$$

where the symbol $\circ_{I_j^c}$ denotes composition of integral operators by integration only over the domain I_j^c complementary to I_j within Γ_j. (Note that $\widetilde{g}_{j+1,j} = g_{j+1.j}$.) Analogously to (1.4), we define the pairing matrix

$$A^I_{ab} := \int_{I_1^c} f_a(x_1)\widetilde{g}^*_{m1}(h_b)(x_1)d\mu_1(x_1) = \int_{I_m^c} \widetilde{g}_{m1}(f_a)(x_m)(h_b)(x_m)d\mu_m(x_m), \tag{1.25}$$

and again assume it to be nonsingular. Once again, making a PLU decomposition of A^I

$$A^I := P^I L^I U^I, \tag{1.26}$$

we may form new bases

$$\widetilde{\psi}^{(1)}_a := \sum_{b=1}^N (P^I L^I)^{-1}_{ab} f_b \qquad \widetilde{\phi}^{(m)}_a := \sum_{a=1}^N (U^I)^{-1}_{ba} h_a \tag{1.27}$$

for the spaces H_1 and H^m, as well as bases for the sequence of dual pairs of spaces obtained by composition with $\widetilde{g}_{j1}$ and $\widetilde{g}^*_{mj}$:

$$\widetilde{\psi}^{(j)}_a := \widetilde{g}_{j1}(\widetilde{\psi}^{(1)}_a), \qquad \widetilde{\phi}^{(j)}_a := \widetilde{g}^*_{mj}(\widetilde{\phi}^{(m)}_a). \tag{1.28}$$

These are mutually dual in the sense that

$$\int_{I_j^c} \widetilde{\psi}^{(j)}_a(x_j)\widetilde{\phi}^{(j)}_b(x_j)d\mu_j(x_j) = \delta_{ab}. \tag{1.29}$$

Define, as before, a set of integral kernels:

$$\widetilde{K}_{ij}(x_i, x_j) := \sum_{a=1}^N \widetilde{\psi}^{(i)}_a(x_i)\widetilde{\phi}^{(j)}_a(x_j) \tag{1.30}$$

and

$$\widetilde{\widetilde{K}}_{ij}(x_i, x_j) := \widetilde{K}_{ij}(x_i, x_j) - \widetilde{g}_{ij}(x_i, x_j), \tag{1.31}$$

where again, $\widetilde{g}_{ij}(x_i, x_j) := 0$ if $i \le j$, and denote by $\widetilde{K}_{ij}$, $\widetilde{g}_{ij}$ and $\widetilde{\widetilde{K}}_{ij}$ the corresponding integral operators mapping $\hat{H}_j \to \hat{H}_i$. The associated $m \times m$ matrix integral operators acting on $\hat{H}$ are similarly denoted $\widetilde{K}$, $\widetilde{g}$ and $\widetilde{\widetilde{K}}$, with

$$\widetilde{\widetilde{K}} := \widetilde{K} - \widetilde{g}. \tag{1.32}$$

Then Soshnikov's theorem states that the resolvent operator R^{χ_I} entering in the definition of the Janossy distribution coincides with the operator $\widetilde{\widetilde{K}}^{\chi_I} := \widetilde{\widetilde{K}} \circ \chi_I$ so constructed.

THEOREM 1:

$$R^{\chi_I} = \check{\check{K}}^{\chi_I}. \tag{1.33}$$

A proof of this theorem is given in [S], based on decomposition of the spaces into various subspaces, and an inductive construction, but it is rather long and intricate. In the following section, a very simple direct proof will be given.

Since no further effort is required, we will actually prove an obvious generalization that reduces to the above result as a special case. Namely, instead of choosing measurable subsets $I_j \subset \Gamma_j$ and their characteristic functions χ_j, we may replace the latter by any set of square integrable functions $\{w_j \in L^2(\Gamma_j, d\mu_j)\}_{j=1\dots m}$, and generalize the definition of the matrix A^I and the functions $\{\widetilde{\psi}^{(j)}, \widetilde{\phi}^{(j)}, \widetilde{g}_{ij}\}$ accordingly to:

$$\begin{aligned} A^w_{ab} &:= \int_{\Gamma_1} f_a(x_1)\widetilde{g}^*_{m1}(h_b)(x_1)(1 - w_1(x_1))d\mu_1(x_1) \\ &= \int_{\Gamma_m} \widetilde{g}_{m1}(f_a)(x_m)(h_b)(x_m)(1 - w_m(x_m))d\mu_m(x_m), \end{aligned} \tag{1.34}$$

with PLU decomposition

$$A^w := P^w L^w U^w, \tag{1.35}$$

where, denoting by $\circ_{v_j}$ the operation of mutiplication by a function v_j in $L^2(\Gamma_j, d\mu_j)$ followed by composition, we replace the definition of $\widetilde{g}_{k,j}$ in (1.24) by

$$\widetilde{g}_{kj} := g_{k,k-1} \circ_{1-w_{k-1}} \circ \dots \circ_{1-w_{j+1}} \circ g_{j+1,j}. \tag{1.36}$$

Then

$$\widetilde{\psi}^{(1)}_a := \sum_{a=1}^{N} (P^w L^w)^{-1}_{ab} f_a \qquad \widetilde{\phi}^{(m)}_a := \sum_{b=1}^{N} (U^w)^{-1}_{ba} h_a, \tag{1.37}$$

define bases for the spaces H_1 and H^m, respectively, while bases for the sequence of dual spaces obtained by composition with $\widetilde{g}_{j,1}$ and $\widetilde{g}^*_{m,k}$, as defined in (1.36), are again given by (1.28), with the dualization pairing given by

$$\int_{\Gamma_j} \widetilde{\psi}^{(j)}_a(x_j)\widetilde{\phi}^{(j)}_b(x_j)(1 - w_j(x_j))d\mu_j(x_j) = \delta_{ab}. \tag{1.38}$$

Replacing the characteristic functions $\chi_I = (\chi_1 \dots \chi_m)$ by the set of weight functions $w := (w_1, \dots w_m)$, the definitions of the integral operators and kernels $\widetilde{K}, \widetilde{g}, \check{\check{K}}$ retain the same form and the theorem becomes

THEOREM 2:

$$R^w = \check{\check{K}}^w := \check{\check{K}} \circ w. \tag{1.39}$$

where

$$R^w := (1 - \check{K}^w)^{-1} \circ \check{K}^w \tag{1.40}$$

is the Fredholm resolvent of

$$\check{K}^w := \check{K} \circ w. \tag{1.41}$$

Such a generalization is of use in multimatrix models, since it allows us to replace, for example, the characteristic functions χ_j on subintervals $I_j \subset \Gamma_j$ by weighted characteristic functions $\sum_{\alpha=1}^{k_j} \kappa_{\alpha,j} \chi_{I_{\alpha,j}}$ over unions of disjoint subintervals $I_j = \cup_{\alpha=1}^{k_j} I_{\alpha,j}$. The corresponding Fredholm determinant (1.23), expanded as a power series in the coefficients $\kappa_{\alpha,j}$ becomes a generating function for the probabilities of having given numbers of eigenvalues within the subintervals, as in the 1-matrix case [TW].

2. Proof of the theorem.

The equality (1.39) may equivalently be written as

$$\check{K}^w \circ \check{\tilde{K}}^w = \check{\tilde{K}}^w - \check{K}^w, \tag{2.1}$$

or, explicitly,

$$\sum_{j=1}^{m} \check{K}_{ij} \circ_{w_j} \check{\tilde{K}}_{jk} = \check{\tilde{K}}_{ik} - \check{K}_{ik} \tag{2.2}$$

(where composition on the right by w is omitted, since the equality will be shown to hold without it). Substituting (1.12) and (1.32) in (2.1), this is equivalent to

$$K \circ_w \widetilde{K} - g \circ_w \widetilde{K} - K \circ_w \widetilde{g} + g \circ_w \widetilde{g} + \widetilde{g} - g = \widetilde{K} - K. \tag{2.3}$$

This relation follows as a consequence of four identities relating the various summands on the left, each of which is easily proved:

$$g \circ_w \widetilde{g} + \widetilde{g} - g = 0 \tag{2.4}$$

$$(K \circ_w \widetilde{K})_{ij} = g_{i1} \circ \widetilde{K}_{1j} + \delta_{i1} \widetilde{K}_{1j} - K_{im} \circ_{1-w_m} \widetilde{g}_{mj} - K_{im} \delta_{mj} \tag{2.5}$$

$$(g \circ_w \widetilde{K})_{ij} = g_{i1} \circ \widetilde{K}_{1j} + \delta_{i1} \widetilde{K}_{1j} - \widetilde{K}_{ij} \tag{2.6}$$

$$(K \circ_w \widetilde{g})_{ij} = K_{ij} - K_{im} \circ_{1-w_m} \widetilde{g}_{mj} - K_{im} \delta_{mj} \tag{2.7}$$

Proof of (2.4)*:* From (1.3) and (1.36) we have, for $k-1 > j > i+1$,

$$\begin{aligned} g_{kj} \circ_{w_j} \widetilde{g}_{ji} &= g_{kj} \circ \widetilde{g}_{ji} - g_{kj} \circ_{1-w_j} \widetilde{g}_{ji} \\ &= g_{kj} \circ \widetilde{g}_{ji} - g_{k,j+1} \circ g_{j+1,j} \circ_{1-w_j} \widetilde{g}_{ji} \\ &= g_{kj} \circ \widetilde{g}_{ji} - g_{k,j+1} \circ \widetilde{g}_{j+1,i}, \end{aligned} \tag{2.8}$$

while for $j = k-1$,

$$\begin{aligned} g_{k,k-1} \circ_{w_{k-1}} \widetilde{g}_{k-1,i} &= g_{k,k-1} \circ \widetilde{g}_{k-1,i} - g_{k,k-1} \circ_{1-w_{k-1}} \widetilde{g}_{k-1,i} \\ &= g_{k,k-1} \circ \widetilde{g}_{k-1,i} - \widetilde{g}_{ki}, \end{aligned} \tag{2.9}$$

and for $j = i+1$

$$\begin{aligned} g_{k,i+1} \circ_{w_{i+1}} \widetilde{g}_{i+1,i} &= g_{k,i+1} \circ \widetilde{g}_{i+1,i} - g_{k,i+1} \circ_{1-w_{i+1}} \widetilde{g}_{i+1,i} \\ &= g_{ki} - g_{k,i+2} \circ \widetilde{g}_{i+2,i}. \end{aligned} \tag{2.10}$$

Summing over j and cancelling all intermediate terms gives the result

$$(g \circ_w \widetilde{g})_{ki} = \sum_{j=i+1}^{k-1} g_{kj} \circ_{w_j} \widetilde{g}_{ji} = g_{ki} - \widetilde{g}_{ki}. \tag{2.11}$$

(Note that this means that the integral operator g^w is the Fredholm resolvent of $\widetilde{g}^w$.)

Proof of (2.5)*:* For $j \neq m$, we have

$$\begin{aligned} K_{ij} \circ_{w_j} \widetilde{K}_{jk} &= K_{ij} \circ \widetilde{K}_{jk} - K_{ij} \circ_{1-w_j} \widetilde{K}_{jk} \\ &= K_{ij} \circ \widetilde{K}_{jk} - K_{i,j+1} \circ g_{j+1,j} \circ_{1-w_j} \widetilde{K}_{jk} \\ &= K_{ij} \circ \widetilde{K}_{jk} - K_{i,j+1} \circ \widetilde{K}_{j+1,k}. \end{aligned} \tag{2.12}$$

Therefore, summing and cancelling the intermediate terms gives

$$\sum_{j=1}^{m} K_{ij} \circ_{w_j} \widetilde{K}_{jk} = K_{i1} \circ \widetilde{K}_{1k} - K_{im} \circ_{1-w_j} \widetilde{K}_{mk}. \tag{2.13}$$

By (1.8) and (1.38), the operators K_{11} and $(\widetilde{K}_{mm} \circ (1-w_m))^*$ act as the identity on the spaces H_1 and H^m, respectively, and hence

$$K_{11} \circ \widetilde{K}_{1k} = \widetilde{K}_{1k}, \qquad K_{im} \circ_{1-w_m} \widetilde{K}_{mm} = \widetilde{K}_{im}. \tag{2.14}$$

It follows from (1.16) and the corresponding relations

$$\widetilde{K}_{ij} = \widetilde{g}_{i1} \circ_{1-w_1} \widetilde{K}_{1m} \circ_{1-w_m} \widetilde{g}_{mj} \tag{2.15}$$

for the $\widetilde{K}_{ij}$'s that, for $i \neq 1$, $j \neq m$,

$$K_{i1} \circ \widetilde{K}_{1k} = g_{i1} \circ \widetilde{K}_{ik}, \qquad K_{im} \circ_{1-w_m} \widetilde{K}_{mj} = K_{im} \circ_{1-w_m} \widetilde{g}_{mj}. \tag{2.16}$$

Combining these relations with (2.13) leads to (2.5).

Proof of (2.6)*:* For $j < i-1$, we have

$$\begin{aligned} g_{ij} \circ_{w_j} \widetilde{K}_{jk} &= g_{ij} \circ \widetilde{K}_{jk} - g_{ij} \circ_{1-w_j} \widetilde{K}_{jk} \\ &= g_{ij} \circ \widetilde{K}_{jk} - g_{i,j+1} \circ g_{j+1,j} \circ_{1-w_j} \widetilde{K}_{jk} \\ &= g_{ij} \circ \widetilde{K}_{jk} - g_{i,j+1} \circ \widetilde{K}_{j+1,k}, \end{aligned} \tag{2.17}$$

while for $j = i-1$,

$$\begin{aligned} g_{i,i-1} \circ_{w_{i-1}} \widetilde{K}_{i-1,k} &= g_{i,i-1} \circ \widetilde{K}_{i-1,k} - g_{i,i-1} \circ_{1-w_{i-1}} \widetilde{K}_{i-1,k} \\ &= g_{i,i-1} \circ \widetilde{K}_{i-1,k} - \widetilde{K}_{ik}, \end{aligned} \tag{2.18}$$

Again, summing, and cancelling the intermediate terms gives:

$$\sum_{j=1}^{m} g_{ij} \circ_{w_j} \widetilde{K}_{jk} = \sum_{j=1}^{i-1} g_{ij} \circ_{w_j} \widetilde{K}_{jk} = g_{i1} \circ \widetilde{K}_{1k} - \widetilde{K}_{ik}, \tag{2.19}$$

which is (2.6) (the case $i = 1$ being trivially satisfied).

Proof of (2.7)*:* For $m > j > k+1$, we have

$$\begin{aligned} K_{ij} \circ_{w_j} \widetilde{g}_{jk} &= K_{ij} \circ \widetilde{g}_{jk} - K_{ij} \circ_{1-w_j} \widetilde{g}_{jk} \\ &= K_{ij} \circ \widetilde{g}_{jk} - K_{i,j+1} \circ g_{j+1,j} \circ_{1-w_j} \widetilde{g}_{jk} \\ &= K_{ij} \circ \widetilde{g}_{jk} - K_{i,j+1} \circ \widetilde{g}_{j+1,k}, \end{aligned} \tag{2.20}$$

while for $j = k+1$,

$$\begin{aligned} K_{i,k+1} \circ_{w_{k+1}} \widetilde{g}_{k+1,k} &= K_{i,k+1} \circ \widetilde{g}_{k+1,k} - K_{i,k+1} \circ_{1-w_{k+1}} \widetilde{g}_{k+1,k} \\ &= K_{ik} - K_{i,k+2} \circ \widetilde{g}_{k+2,k}, \end{aligned} \tag{2.21}$$

Again, summing and cancelling the intermediate terms gives:

$$\sum_{j=1}^{m} K_{ij} \circ_{w_j} \widetilde{g}_{jk} = \sum_{j=k+1}^{m} K_{ij} \circ_{w_j} \widetilde{g}_{jk} = K_{ik} - K_{im} \circ_{1-w_m} \widetilde{g}_{mk}, \tag{2.22}$$

which is (2.7) (the case $j = m$ again being trivially satisfied).

Combining the four relations (2.4), (2.5), (2.6) and (2.7) gives (2.3), hence proving the theorem.

Remark. Although this theorem has been formulated with respect to probability measures and Hilbert spaces of functions on them, it may clearly be extended to more general cases involving, e.g., complex measures, provided all the integrals appearing are convergent, and the related integral operators and their composites are well-defined.

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